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Exact Solution for a Multiphoton Generalized Jaynes-Cummings Model with Inverse Operators

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Abstract A new exactly solvable multiphoton generalized Jaynes-Cummings model is presented, whose Hamiltonian is related to the inverse of field mode creation and annihilation operators. Then we use supersymmetric unitary operators to diagonalize the Hamiltonian above and obtain their energy spectra and eigenstates. In addition, its pseudo-invariant eigenoperator is found as well, directly leading to the corresponding energy-level gap.

Keywords Supersymmetric transformation generators · Pseudo-invariant eigenoperator · Inverse operators · Multiphoton Jaynes-Cummings model

1 Introduction

The interaction of atoms with the electromagnetic field is attracting much attention from many researchers in the fields of quantum optics [1–3]. The simplest physical situation can be described by the well-known Jaynes-Cummings (JC) model [4], where the interaction of a single two-level atom with one cavity field mode near resonance is studied under the condition of the rotating-wave approximation and exhibits many interesting quantum effects. In a series of papers [5–8], the basic JC model has been extended to the field intensity-dependent coupling case and the case of the multiphoton interaction between field and atoms, and the corresponding behavior of the field state statistics has been studied. As is well known, the multiphoton JC model, previously treated by Buck and Sukumar [5, 6], is the simplest exactly solvable generalization of the Jaynes-Cummings model. They have found the exact solutions of the Heisenberg equations for the field and atomic operators. For a single-mode

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electromagnetic field interacting with a two-level atom via an k-photon process, its Hamiltonian under the rotating wave approximation reads ($\hbar = 1$)

$$H = \Omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + \Gamma_k(\sigma_+ a^k + a^{\dagger k}\sigma_-), \tag{1}$$

where Γ_k denotes the multiphoton atom–field coupling coefficient and k is the photon multiple. Ω and ω_0 are the frequencies of the cavity mode and transition frequency of the atom, respectively. In the case of exact resonance, $\omega_0 = k\Omega$. σ_z, σ_+ and σ_- are atomic pseudospin operators, while $a^\dagger(a)$ is photon creation (annihilation) operator, satisfying $[a, a^\dagger] = 1$. Subsequently, the statistical properties of the radiation in this model have been studied by Singh [9]. This model can also be solved in the framework of the path-integral formalism by Bužek [10]. Furthermore, in such a model, second-order squeezing and fourth-order squeezing are exhibited, respectively, in Refs. [11] and [12]. Recently, Shen group have used the invariant theory to exact solution of the time-dependent multiphoton JC model [13].

In the present work, we present another new exactly solvable multiphoton generalized Jaynes-Cummings model, whose Hamiltonian is related to the inverse of field mode creation and annihilation operators

$$\mathcal{H} = \Omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + g_k[\sigma_+(a^\dagger)^{-k} + a^{-k}\sigma_-], \tag{2}$$

where g_k also denotes the multiphoton atom-field coupling coefficient, $(a^\dagger)^{-1}$ and a^{-1}, respectively, represent the inverse of creation operator $a^\dagger$ and annihilation operator a, i.e., $aa^{-1} = (a^\dagger)^{-1}a^\dagger = 1$, but $a^{-1}a \neq 1, a^\dagger(a^\dagger)^{-1} \neq 1$ because $a|0\rangle = 0$, $\langle 0|a^\dagger = 0$ as point out in Refs. [14] and [15]. This model may describe some nonlinear processes in laser physics. To our best knowledge, this multiphoton generalized JC model has not been clearly discussed in the literature before. Our main task is to exactly solve the energy spectra and eigenstates of the above system in (2) by virtue of the supersymmetric unitary transformation and search for the so-called pseudo-invariant eigenoperator. In the following section, we firstly outline some commutators about the inverse operators a^{-1} and $(a^\dagger)^{-1}$, which shows that a^{-1} behaves like a creator, whereas $(a^\dagger)^{-1}$ behaves like an annihilator. In Sect. 3, we use supersymmetric unitary transformation to diagonalize the Hamiltonian in (2) and obtain the enery-level gap. We devote Sect. 4 to finding so-called pseudo-invariant eigenoperator, directly leading to the energy-level gap for the above system. Section 5 concludes our letter with a summary and some discussion.

2 Some Commutators About a^{-1} and $(a^\dagger)^{-1}$

To begin with, we briefly outline some commutators about the inverse operators a^{-1} and $(a^\dagger)^{-1}$. Recall that in Ref. [15], the following Fock representations of a^{-1} and $(a^\dagger)^{-1}$ have been derived via the coherent state approach as follows

$$a^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}|n+1\rangle\langle n|, \qquad (a^\dagger)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}|n\rangle\langle n+1| = (a^{-1})^\dagger, \tag{3}$$

where $|n\rangle$ is the eigenstate of $N = a^\dagger a$ with $|n\rangle = a^{\dagger n}|0\rangle/\sqrt{n!}$. As a result,

$$a^{-1}a = 1 - |0\rangle\langle 0| = a^\dagger(a^\dagger)^{-1}, \qquad [(a^\dagger)^{-1}, a^\dagger] = |0\rangle\langle 0|. \tag{4}$$

Further, using (3) and $a|n\rangle = \sqrt{n}|n-1\rangle$, it is deduced that

$$\left[a, (a^\dagger)^{-1}\right] = \sum_{n=2}^{\infty} \frac{-1}{\sqrt{n(n-1)}}|n-2\rangle\langle n| = -(a^\dagger)^{-2} = \frac{\partial}{\partial a^\dagger}(a^\dagger)^{-1}. \tag{5}$$

Due to (3), ones have

$$\left[(a^\dagger)^{-1}, a^\dagger a\right] = |0\rangle\langle 0|a + a^\dagger(a^\dagger)^{-2} = |0\rangle\langle 1| + (1 - |0\rangle\langle 0|)(a^\dagger)^{-1} = (a^\dagger)^{-1}, \tag{6}$$

which shows the following commutators

$$[a^{-1}, N] = -a^{-1}, \qquad \left[(a^{-1})^\dagger, N\right] = (a^{-1})^\dagger. \tag{7}$$

From Ref. [15],

$$\left[a, f[(a^\dagger)^{-1}]\right] = \frac{\partial}{\partial a^\dagger} f\left[(a^\dagger)^{-1}\right], \qquad \left[a^\dagger, f(a^{-1})\right] = -\frac{\partial}{\partial a} f(a^{-1}), \tag{8}$$

it is easily obtained that

$$\left[(a^\dagger)^{-n}, N\right] = n(a^\dagger)^{-n}, \qquad [a^{-n}, N] = -na^{-n}, \tag{9}$$

which is just negative power generalization of the well-known commutators

$$[a^{\dagger n}, N] = -na^{\dagger n}, \qquad [a^n, N] = na^n. \tag{10}$$

It is seen from (3) and (9) that the inverse operators a^{-1} behaves like a creator, whereas $(a^\dagger)^{-1}$ behaves like an annihilator.

3 Supersymmetric Unitary Transformation for $\mathcal{H}$

Since there exists a supersymmetry in some JC models [16–18], here we obtain the exact solution of $\mathcal{H}$ in (2) from supersymmetric quantum mechanics viewpoint. For this purpose, considering that σ_z, σ_+ and σ_- are atomic pseudospin operators, expressed as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{11}$$

we first define the supersymmetric transformation generators as follows

$$N \equiv a^\dagger a + \frac{k-1}{2}\sigma_z + \frac{1}{2} = \begin{pmatrix} a^\dagger a + \frac{k}{2} & 0 \\ 0 & aa^\dagger - \frac{k}{2} \end{pmatrix}, \tag{12}$$

$$Q \equiv a^{-k}\sigma_- = \begin{pmatrix} 0 & 0 \\ a^{-k} & 0 \end{pmatrix}, \qquad Q^\dagger \equiv \sigma_+(a^\dagger)^{-k} = \begin{pmatrix} 0 & (a^\dagger)^{-k} \\ 0 & 0 \end{pmatrix}, \tag{13}$$

and

$$N' \equiv \{Q^\dagger, Q\} = \begin{pmatrix} (a^\dagger)^{-k}a^{-k} & 0 \\ 0 & a^{-k}(a^\dagger)^{-k} \end{pmatrix}, \tag{14}$$

where { } stands for an anticommuter. Using (9) and the commutative relation

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm, \qquad [\sigma_+, \sigma_-] = \sigma_z, \tag{15}$$

it is easily seen that $(N, N', Q^\dagger, Q)$ form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.,

$$\begin{aligned} &Q^2 = Q^{\dagger 2} = 0, \qquad [Q^\dagger, Q] = N'\sigma_z, \qquad (Q^\dagger - Q)^2 = -N', \\ &[N, Q] = Q, \qquad [N, Q^\dagger] = -Q^\dagger, \qquad [N, N'] = 0, \\ &[Q, \sigma_z] = 2Q, \qquad [Q^\dagger, \sigma_z] = -2Q^\dagger, \qquad \{Q, \sigma_z\} = \{Q^\dagger, \sigma_z\} = 0. \end{aligned} \tag{16}$$

In terms of (12) and (13), equation (2) can be rewritten as

$$\mathcal{H} = \Omega N + \frac{1}{2}(\Omega - \Delta)\sigma_z + g_k(Q + Q^\dagger) - \frac{\Omega}{2}, \tag{17}$$

where $\Delta = k\Omega - \omega_0$ is a detuning quantity.

Now, by the aid of supersymmetric generators mentioned above, we construct the supersymmetric unitary transformation operators to diagonalize the Hamiltonian in (17). The supersymmetric unitary transformation operator is defined as the following

$$T = \exp\left\{-\frac{\theta}{2} N'^{-1/2}(Q^\dagger - Q)\right\}, \tag{18}$$

in which $N'^{1/2}$ is

$$N'^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{(a^\dagger)^{-k} a^{-k}}} & 0 \\ 0 & \frac{1}{\sqrt{a^{-k}(a^\dagger)^{-k}}} \end{pmatrix}. \tag{19}$$

By using $(a^\dagger)^{-1} f[a^{-1}(a^\dagger)^{-1}] = f[(a^\dagger)^{-1}a^{-1} + 1](a^\dagger)^{-1}$, $a^{-1} f[(a^\dagger)^{-1}a^{-1}] = f[a^{-1}(a^\dagger)^{-1}]a^{-1}$, we easily get

$$[N'^{-1/2}, Q^\dagger] = [N'^{-1/2}, Q] = 0. \tag{20}$$

Therefore, from (16) and (20), equation (18) can be expanded to the following form

$$T = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) N'^{-1/2}(Q^\dagger - Q), \tag{21}$$

so

$$T^{-1} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) N'^{-1/2}(Q^\dagger - Q) = T^\dagger. \tag{22}$$

Using (16), (21) and (22), it then follows that

$$\begin{aligned} &T^{-1} N T = N + \frac{1}{2}\sin\theta N'^{-1/2}(Q^\dagger + Q) + \sin^2\left(\frac{\theta}{2}\right)\sigma_z, \\ &T^{-1}(Q^\dagger + Q)T = \cos\theta(Q^\dagger + Q) + \sin\theta N'^{-1/2}[Q^\dagger, Q], \\ &T^{-1}\sigma_z T = \cos\theta\sigma_z - \sin\theta N'^{-1/2}(Q^\dagger + Q). \end{aligned} \tag{23}$$

Based on the above equations, equation (17) can be easily performed by T^{-1}

$$\begin{aligned}\mathcal{H}' &= T^{-1}\mathcal{H}T \\ &= \Omega N - \frac{\Omega}{2} + \left(\frac{\Delta}{2}\sin\theta N'^{-1/2} + g_k\cos\theta\right)(Q+Q^\dagger) \\ &\quad + \left[\frac{1}{N'^{1/2}}\left(\frac{\Omega}{2} - \frac{\Delta}{2}\cos\theta + g_k\sin\theta\right)\right]\frac{1}{N'^{1/2}}[Q^\dagger, Q].\end{aligned} \tag{24}$$

Formally, if we annihilate the second term of (24) by letting

$$\frac{1}{N'^{1/2}}\tan\theta = -\frac{2g_k}{\Delta}. \tag{25}$$

We can obtain the diagonalized Hamiltonian as follows

$$\mathcal{H}' = \Omega N - \frac{\Omega}{2} + \left(\frac{\Omega}{2} - \frac{\Delta}{2\cos\theta}\right)\sigma_z. \tag{26}$$

Note that (25) should be understood in the sense of eigenvalue and eigenvalue equation for the operator N', which is

$$\begin{aligned}|\Phi'\rangle_1 &= |n\rangle\otimes|+\rangle = \begin{pmatrix}|n\rangle\\0\end{pmatrix}, \qquad N'|\Phi'\rangle_1 = \frac{n!}{(n+k)!}|\Phi'\rangle_1, \\ |\Phi'\rangle_2 &= |n+k\rangle\otimes|-\rangle = \begin{pmatrix}0\\|n+k\rangle\end{pmatrix}, \qquad N'|\Phi'\rangle_2 = \frac{n!}{(n+k)!}|\Phi'\rangle_2,\end{aligned} \tag{27}$$

where $\sigma_z|\pm\rangle = \pm1|\pm\rangle$. From (25), let $N' = \frac{n!}{(n+k)!}$, we get

$$\cos\theta = -\frac{\Delta}{[\omega^2+\Delta^2]^{1/2}}, \tag{28}$$

where $\omega = 2g_k[n!/(n+k)!]^{1/2}$ is Rabi frequency. It is easily seen that

$$N|\Phi'\rangle_i = \left(n+\frac{k}{2}\right)|\Phi'\rangle_i, \quad i=1,2. \tag{29}$$

From the above analysis, the corresponding eigenvalues and eigenstates of $\mathcal{H}$ in (2) are given by, respectively,

$$\begin{aligned}E_1 &= \Omega\left(n+\frac{k}{2}\right) + \frac{1}{2}[\omega^2+\Delta^2]^{1/2}, \\ |\Phi_1\rangle &= T|\Phi'\rangle_1 = \left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)N'^{-1/2}(Q^\dagger - Q)\right]\begin{pmatrix}|n\rangle\\0\end{pmatrix}, \\ &= \cos\left(\frac{\theta}{2}\right)\begin{pmatrix}|n\rangle\\0\end{pmatrix} + \sin\left(\frac{\theta}{2}\right)\begin{pmatrix}0\\|n+k\rangle\end{pmatrix},\end{aligned} \tag{30}$$

and

$$E_2 = \Omega\left(n+\frac{k}{2}\right) - \frac{1}{2}[\omega^2+\Delta^2]^{1/2},$$

$$
\begin{aligned}
|\Phi_2\rangle = T|\Phi'\rangle_2 &= \left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) N'^{-1/2}(Q^\dagger - Q)\right]\begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix}, \\
&= \cos\left(\frac{\theta}{2}\right)\begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix} - \sin\left(\frac{\theta}{2}\right)\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}.
\end{aligned}
\tag{31}
$$

It should be pointed out that the state $\Phi_3 = |n\rangle \otimes |-\rangle = \binom{0}{|n\rangle}$, $(n \leq k-1)$, which is not included in the above discussion, is another eigenstate of $\mathcal{H}$.

4 Pseudo-invariant Eigenoperator of $\mathcal{H}$

In this section, we turn our attention to search for the pseudo-invariant eigen-operator $\hat{O}_e$ of $\mathcal{H}$ in terms of supersymmetric generators (see (17)). The so-called pseudo-invariant eigen-operator (PIEO) $\hat{O}_e$, first proposed by Fan et al. [19] via combining the Schrödinger operator and the Heisenberg equation of motion, should satisfy the following equation

$$
[[\hat{O}_e, \mathcal{H}], \mathcal{H}]|\psi\rangle_i = (\Delta E)^2 \hat{O}_e|\psi\rangle_i \tag{32}
$$

in the eigenvector space $|\psi\rangle_i$ of conservation quantities of the dynamic system, where ΔE is the energy-level gap of the dynamic Hamiltonian, which may explore the energy-level gap of some JC models with convenience and directness [20, 21].

Next, according to Fan's pseudo-invariant eigenoperator theory, we assume that the PIEO of Hamiltonian in (17) possesses the form

$$
\hat{O}_e = \alpha(Q^\dagger + Q) + \beta\sigma_z \tag{33}
$$

where α and β are undermined constants. Using the relations in (16) and (17), we calculate

$$
i\frac{d}{dt}\hat{O}_e = [\hat{O}_e, \mathcal{H}] = (\alpha\Omega + 2\beta g_k)(Q^\dagger - Q). \tag{34}
$$

Further calculation shows

$$
\left(i\frac{d}{dt}\right)^2 \hat{O}_e = [(\alpha\Omega + 2\beta g_k)(Q^\dagger - Q), \mathcal{H}] = (\alpha\Omega + 2\beta g_k)[\Delta(Q^\dagger + Q) + 2g_k N'\sigma_z]. \tag{35}
$$

By acting the two sides of (35) on the eigenstates $|\Phi'\rangle_i$ of N' in (27), we have

$$
\left(i\frac{d}{dt}\right)^2 \hat{O}_e|\Phi'\rangle_i = (\alpha\Omega + 2\beta g_k)\left[\Delta(Q^\dagger + Q) + \frac{\omega^2}{2g_k}\sigma_z\right]|\Phi'\rangle_i, \tag{36}
$$

which is a form like (32). From (33) and (36), we obtain

$$
\alpha = \frac{2g_k\Delta}{\omega^2}\beta. \tag{37}
$$

Thus in the Hilbert space spanned by the eigenstates $|\Phi'\rangle_i$, we can determine the expression of $\hat{O}_e$

$$
\hat{O}_e = \frac{2g_k\Delta}{\omega^2}\beta(Q^\dagger + Q) + \beta\sigma_z, \tag{38}
$$

which is called a pseudo-invariant eigenoperator of (2). Substituting (38) and (36) into (32), we get

$$\lambda = \omega^2 + \Delta^2 \tag{39}$$

and further obtain the energy-level gap for this system as

$$\sqrt{\lambda} = \sqrt{\omega^2 + \Delta^2} \tag{40}$$

which cioncides with the eigen-energy of $\mathcal{H}$ in (30) and (31), i.e.

$$\sqrt{\lambda} = \sqrt{\omega^2 + \Delta^2} = E_2 - E_1. \tag{41}$$

From the above discussion, in order to get the PIEOs, the key point is to find the conservative quantities of the corresponding model.

5 Conclusions

In summary, we have introduced supersymmetric unitary operators to directly diagonalize the new multiphoton generalized Jaynes-Cummings model whose Hamiltonian is related to the inverse of field mode creation and annihilation operators. Besides, based on the JC mode described by the supersymmetric generators, we find the pseudo-invariant eigen-operator in terms of supersymmetric generators. It is worth mentioning that the PIEO method may more simply obtain the energy level of some generalized JC models. There is no doubt that this method will further enrich the approaches for solving the eigenvalue equations of quantum mechanics.

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